

Do Neutrino Wave Functions Overlap and Does it Matter?

CHENG-HSIEN LI & YONG-ZHONG QIAN

*School of Physics and Astronomy, University of Minnesota
116 Church Street S.E., Minneapolis, MN 55455, USA*

Studies of neutrinos commonly ignore anti-symmetrization of their wave functions. This implicitly assumes that either spatial wave functions for neutrinos with approximately the same momentum do not overlap or their overlapping has no measurable consequences. We examine these assumptions by considering the evolution of three-dimensional neutrino wave packets (WPs). We find that it is perfectly adequate to treat accelerator and reactor neutrinos as separate WPs for typical experimental setup. While solar and supernova neutrinos correspond to overlapping WPs, they can be treated effectively as non-overlapping for analyses of their detection.

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1 Introduction

Quantum particles can be treated as wave packets (WPs). The use of neutrino WPs clarified some conceptual confusion [1] from the use of plane waves to discuss neutrino oscillations and introduced a few oscillation-suppressing terms in the flavor transformation probabilities. For that problem, one-dimensional (1D) consideration suffices as only the longitudinal evolution of neutrino WPs is of concern. Nevertheless, the simplified 1D description is not a full account of WP propagation in the 3D space.

According to Heisenberg's uncertainty principle, a particle localized in a finite spatial region has intrinsic momentum uncertainty that causes the WP to spread over time. The longitudinal spreading of a neutrino WP is suppressed by the tiny neutrino mass [2] and can be neglected. In contrast, the transverse size of the WP increases with the time of travel t as $(\Delta k_{\perp}/k_0)t$, where Δk_{\perp} and k_0 are the transverse momentum uncertainty and the average momentum of the WP, respectively. Limited by the speed of light, the WP asymptotically evolves into a spherical shape that subtends a constant angle from its initial position, as depicted in Fig. 1a.

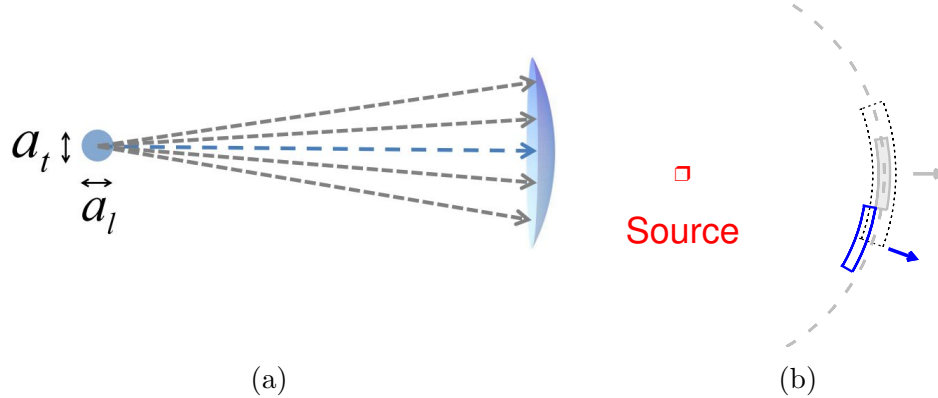


Figure 1: (a) Spreading of a massless neutrino WP with initial position uncertainties a_l and a_t . The blue dashed arrow shows the classical path. (b) Criteria for overlap. The blue and gray strips represent the 90%-probability volumes of two WPs.

The transverse spreading potentially allows neutrino WPs emitted in slightly different directions but with approximately the same momentum to overlap. If such overlap occurs, the indistinguishability of neutrinos would require a formal many-particle treatment. Here we describe an approximate approach to quantify the overlap based on the evolution of 3D Gaussian WPs. We apply this approach to accelerator, reactor, solar and supernova neutrinos and find no and severe overlap for the former and latter two sources, respectively. However, the overlap has no measurable consequences for practical detection of solar and supernova neutrinos.

2 Estimating Overlap of 3D WPs

As the tiny neutrino mass introduces negligible longitudinal WP spreading, we assume massless neutrinos to focus on the transverse spreading. We consider a Gaussian WP with momenta closely centered around the mean value $\vec{k}_0 = k_0 \hat{z}$. An initial WP with the position widths shown in Fig. 1a can be expressed as

$$\Psi(\vec{r}, 0) = \frac{1}{(2\pi)^{3/4} a_t a_l^{1/2}} \exp\left(-\frac{\rho^2}{4a_t^2} - \frac{z^2}{4a_l^2} + ik_0 z\right), \quad (1)$$

where $\rho \equiv \sqrt{x^2 + y^2}$. At time $t > 0$, the above WP evolves into [3]

$$\Psi(\vec{r}, t) \propto \frac{1}{z} \exp\left\{-\frac{[z + \rho^2/(2z) - t]^2}{4a_l^2} - \frac{\rho^2}{z^2(k_0 a_t)^{-2}} + ik_0 \left(z + \frac{\rho^2}{2z} - t\right)\right\}. \quad (2)$$

With $r \approx z + \rho^2/(2z)$, the corresponding probability density can be expressed in spherical coordinates as $|\Psi(\vec{r}, t)|^2 = R(r, t)\Theta(\theta)$, where

$$R(r, t) \propto \frac{1}{r^2} \exp\left[-\frac{(r - t)^2}{2a_l^2}\right], \quad \Theta(\theta) \propto \exp\left[-\frac{\theta^2}{2 \cdot (2k_0 a_t)^{-2}}\right]. \quad (3)$$

For estimating the overlap of WPs, we define the 90%-probability volume by $t - 2a_l < r < t + 2a_l$ and $0 \leq \theta \lesssim 1.22(k_0 a_t)^{-1}$, which corresponds to the 95%-probability regions of the radial and angular distributions.

We regard two WPs as overlapping if their 90%-probability volumes intersect (see Fig. 1b) and if their energies are the same within the intrinsic uncertainty ΔE . This defines the emission time window $\tau \sim \mathcal{O}[(\Delta E)^{-1}]$ and solid angle $\Delta\Omega_{\text{overlap}} \sim \mathcal{O}[(k_0 a_t)^{-2}]$ (see Fig. 1b) for the WPs of concern. With a differential production rate $d^2\Phi/dE_\nu d\Omega$ for the source, the number of WPs expected to overlap with a reference WP is

$$\eta = \tau \frac{d^2\Phi}{dE_\nu d\Omega} (\Delta E) (\Delta\Omega_{\text{overlap}}) \sim \frac{d^2\Phi}{dE_\nu d\Omega} \frac{96\pi}{(k_0 a_t)^2} \sim 96\pi \frac{d^2\Phi}{dE_\nu d\Omega} \left(\frac{\Delta k_\perp}{k_0}\right)^2, \quad (4)$$

where $\Delta k_\perp \sim a_t^{-1}$ is the transverse momentum uncertainty of the WPs. Because the WPs have sharply-peaked momentum distributions, $\Delta k_\perp \ll k_0$ and

$$\eta \ll 96\pi \frac{d^2\Phi}{dE_\nu d\Omega}. \quad (5)$$

The numerical factor in Eqs. (4) and (5) comes from generous estimates of ΔE , τ , and $\Delta\Omega_{\text{overlap}}$. We will see that this factor does not affect our results below.

We give characteristic parameters for accelerator, reactor, solar, and supernova neutrinos along with the corresponding estimates of $\eta(k_0 a_t)^2$ in Table 1. It can be seen that accelerator and reactor neutrinos can be safely treated as non-overlapping WPs. However, for reasonable guesses of $k_0 a_t$, solar and supernova neutrinos correspond to overlapping WPs.

Table 1: Characteristics of various neutrino sources

Source	$d^2\Phi/dE_\nu d\Omega$ ($\text{MeV}^{-1} \text{s}^{-1} \text{sr}^{-1}$)	$\eta(k_0 a_t)^2$	D (m)	E_ν (MeV)	HBT setup
accelerator	$\sim 10^{18}$	~ 0.1	$\sim 10^6$	$\sim 10^3$	No
reactor	$\sim 10^{19}$	~ 1	$\sim 10^3\text{--}10^5$	~ 1	No
sun	$\sim 10^{32}\text{--}10^{38}$	$\sim 10^{13}\text{--}10^{19}$	$\sim 10^{11}$	$\sim 0.1\text{--}10$	No
supernova	$\sim 10^{51}\text{--}10^{55}$	$\sim 10^{32}\text{--}10^{36}$	$\sim 10^{20}$	~ 10	Yes

3 Physics of Quantum WPs: Overlapping or Not

When WPs do not overlap, one might ask how the 3D WP treatment can be reconciled with the picture of bullet-like particles, which is commonly assumed for analyzing neutrino detection. As the transverse size of the WP can easily become macroscopically large, any microscopic detection process only “sees” a 1D neutrino wave train weighted by a direction-dependent amplitude, the square of which is $\Theta(\theta)$ in Eq. (3). Because of the spherical wave front in Eq. (2), the observed effective plane-wave momentum points in the same direction as that defined in the classical sense. In addition, without interference among the WPs, the observed flux is a sum over WPs emitted in different directions. As $\Theta(\theta)$ is normalized, this flux is the same as the particle number flux from a source emitting bullet-like particles.

When WPs overlap, formally neutrinos should be described by an anti-symmetric many-particle wave function. However, if among the overlapping WPs, only one neutrino is detected at a time, it can be shown that the interference terms in the one-particle detection probability are proportional to the inner products of the one-particle states. Although these states overlap in both position and momentum spaces at the detector, they are in fact orthogonal because their production processes are spatially separated at the source. Consequently, the one-particle detection rate is not affected by the overlap of WPs.

If more than one neutrino can be detected simultaneously, then the Hanbury Brown and Twiss (HBT) effect may take place. For fermions such as neutrinos, the HBT effect causes the detected events to “anti-bunch” if certain criteria are met [4]. For this to occur, the WPs from production points a and b must overlap at detection points c and d as shown in Fig. 2. More specifically, the geometric condition

$$E_\nu r_{ab,\perp} r_{cd,\perp} / D \lesssim 1 \quad (6)$$

must be satisfied and the temporal separation between the two detected events must be less than the coherence time of the source. In Eq. (6), $r_{ab,\perp}$ ($r_{cd,\perp}$) is the distance between points a and b (c and d) in the direction perpendicular to the source-detector axis and D is the source-detector distance. When the above conditions are realized,

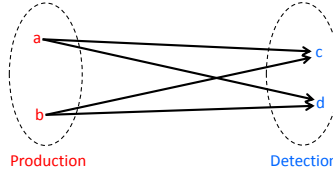


Figure 2: Overlapping WPs and the HBT effect. Ambiguity in pairing the production and detection processes gives rise to the HBT effect.

the two neutrinos would be rendered in the same phase space cell by the detection processes. Therefore, such joint detection would be suppressed by the Pauli exclusion principle, which accounts for the anti-bunching.

Using the parameters given in Table 1, we find that the geometric condition in Eq. (6) is only satisfied for neutrinos from a Galactic supernova. However, even with a megaton detector, the expected number of $\bar{\nu}_e + p \rightarrow n + e^+$ events from overlapping WPs is $\sim 10^{-14}(\text{kpc}/D)^2$, which is simply too small to show the HBT effect.

4 Summary

Based on the evolution of 3D Gaussian WPs, we have estimated the potential overlap among WPs for accelerator, reactor, solar and supernova neutrinos. We find that no overlap occurs for the former two sources and that the overlap for the latter two does not have measurable consequences. For analyzing detection of neutrinos from the above four sources, it is appropriate to treat neutrinos as separate WPs.

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